A simple course in the theory of relativity

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August 19, 2017

Abstract

This course gives a concise but simple understandable elaboration of the special theory of relativity close to how Albert Einstein did it.

1 The Special theory of relativity

1.1 Slowdown of clocks and contractions in the direction of velocity

Assume two non-rotating[1] reference systems K and K' (relatively) move with velocity v along parallel x-axes as in Figure 1. A point event P has, relatively to K, a location given by the coordinates (x, y, z) and time t. The same event has, relatively to K', the coordinates (x', y', z') and time t'. Assume, for simplicity, the y-axes (of K and K') coincide for $t = 0$ and $t' = 0$ (i.e. the clocks for K and K' are synchronized and set to zero when the y-axes coincide).

Assume a light beam starts from the origin of K' along its y-axis at $t' = 0$ (i.e. when the y-axes of K and K' coincide). Let us call this event P_1 which for both K and K' has x-coordinate 0 and time 0 (for simplicity: $(x, t) = (x', t') = (0, 0)$).

Let us call P_2 the incident that the light beam reaches a point p' on the y'-axis. Recall that the velocity c of light is a universal constant (the same relatively to both K and K'). The point event P_2 has K- and K'-coordinates respectively $(x, t) = (vt, t)$ and $(x', t') = (ct', t')$. An observer moving with K' will observe the light has traveled a distance equal to $x' = \sqrt{(ct)^2 - (vt)^2}$. This gives that $t' = t\sqrt{1 - v^2/c^2}$ So an observer moving with K will see that a clock moving with K' slows down with a factor $\sqrt{1 - v^2/c^2}$

[1] relative to the fix starts

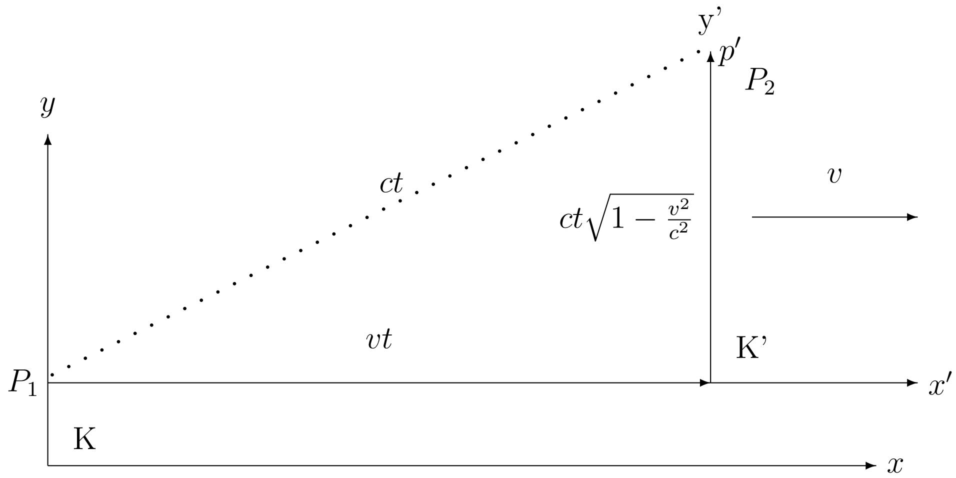

Figure 1: Two events, P_1 and P_2, as viewed by an observer in rest relative to the coordinate system K. P_1 is the start of a light beam when the y-axes coincide. P_2 is the event that the light beam reaches a point p' on the y-axis of K'.

An observer moving with K' will see that the origin of K has moved a distance of $x\sqrt{1-v^2/c^2}$ during this time (i.e. he will see a contraction of objects moving with K in the velocity direction with a factor $\sqrt{1-v^2/c^2}$). K will observe the same contraction for K'.

1.2 The Lorenz transformation

Figure 2 illustrates two moving coordinate systems, K and K', in a similar situation as above (for Figure 1). The relative velocity v is along the x-axes and the clocks of K and K' are synchronized when the y-axes coincide. P_1 is the event that the origin of K' passes the y-axis of K (i.e. it has for both K and K' coordinates $(x,t) = (x',t') = (0,0)$. P_2 is the event that a point p' on the x-axis of K' passes above a point p on the x-axis of K.

Let (x,t) and (x',t') be the coordinates of the event P_2 relatively to K and K' respectively. Following the arguments in Section 1.1 about contractions in the x-direction we get that

$$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}} \tag{1}$$

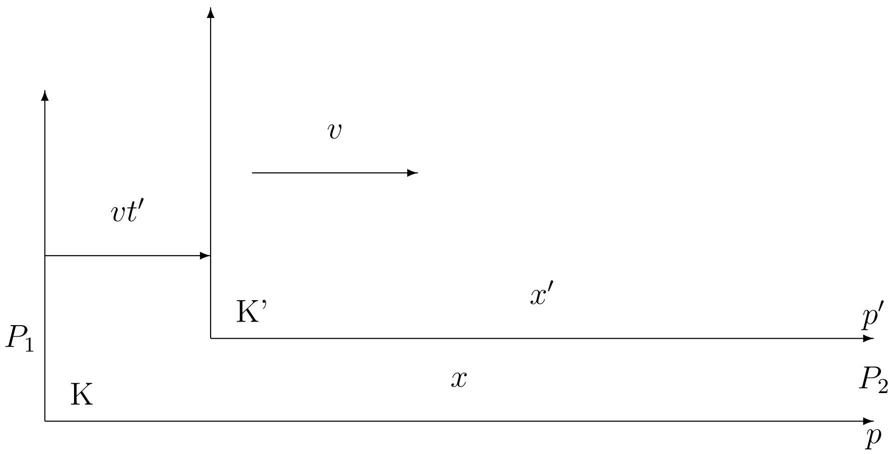

Figure 2: Two events, P_1 and P_2, as viewed by an observer in rest relative to the coordinate system K. P_1 is the passage of the origin of K' cross the y-axis of K. P_2 is the event that the point p', with fixed position relatively to K', passes a similar fixed point on the x-axis of K.

and $vt' = x\sqrt{1 - v^2/c^2} - x'$. This gives:

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}} \tag{2}$$

Symmetry simply gives:

$$y' = y \tag{3}$$

$$z' = z \tag{4}$$

Equations 1, 2, 3 and 4 are called the Lorenz transformation.

1.3 Energy is equivalent to mass ($E = mc^2$)

The fact that light has momentum indicates that mass and energy are equivalent entities. Light in a box with perfect mirror walls behaves like containing mass requiring force to accelerate. Consider, similarly to the light beam of Figure 1, an object with mass m moving relatively to a coordinate system K' along its y-axis with velocity u. The y-component of its velocity relatively to K is $u\sqrt{1 - v^2/c^2}$. Assuming conservation of momentum (mass times velocity), this gives that the mass relatively to K has increased with a factor of $1/\sqrt{1 - v^2/c^2}$.

Assume an object with initial ("rest") mass m which is equivalent to the amount of energy E, and assume it has a small velocity ($v \ll c$). The total energy is $E/\sqrt{1-v^2/c^2}$ which by Taylor expansion gives:

$$\frac{E}{\sqrt{1-v^2/c^2}} = E - \frac{E\,v^2}{2\,c^2} + \frac{3\,E\,v^4}{8\,c^4} + \mathrm{O}(v)^5 \tag{5}$$

However, Newtonian mechanics is proved valid for small velocities. This gives:

$$\frac{1}{2}E\frac{v^2}{c^2} = \frac{1}{2}mv^2 \tag{6}$$

This gives:

$$E = mc^2 \tag{7}$$

2 The general theory of relativity

Given two point events P_1 and P_2 with respective position vectors $\mathbf{r}_1 = (x_1, y_1, z_1, t_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2, t_2)$ relatively to the inertial coordinate system K. Let $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ denote the displacement vector from P_1 to P_2 (relatively to K). Given the notation in component form: $\mathbf{r} = (x, y, z, t)$. Let $r' = (x', y', z', t')$ similarly denote the displacement vector from P_1 to P_2 relatively to another coordinate system K'.

The Lorentz transformation (Equations 1, 3, 4 and 2) give:

$$x^2 + y^2 + z^2 - (ct)^2 = x'^2 + y'^2 + z'^2 - (ct')^2 \tag{8}$$

(Compare this with Figure 1).

...to be continued...